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Weak normalisation and the power series rings

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Abstract

It is shown that if a ring A is weakly normal in an overring B then so is $A[[X]]$ (resp. $A[X]$) in $B[[X]]$ (resp. $B[X]$). It is also shown that every higher derivation of a ring A extends to the weak normalisation of A in its total quotient ring.

The aim of this note is to prove a result on the weak normalisation of $A[[X]]$ (resp. $A[X]$) in $B[[X]]$ (resp. $B[X]$) for the given extension $A \subset B$ of rings, where X is an indeterminate (Theorem 6 and Corollary 8).

Using this result we show that every higher derivation of a ring A extends to the weak normalisation of A in its total quotient ring (Theorem 10).

1. Weak normalisation and the power series rings

By a ring we mean a commutative ring with unity.

For nonnegative integers n and i, let $\binom{n}{i}$ denote the binomial coefficient $n!/(n-i)!i!$. By convention $\binom{n}{i} = 0$ for $n < i$.

We recall the following characterisation of the weak normalisation from [5].

Definition 1. Let $A \subset B$ be an extension of rings. Then the weak normalisation ${}_B^*A$ of A in B is the largest subring C of B containing A such that C is integral over A, the

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induced map $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$ is a bijection and for all $\mathfrak{q} \in \mathrm{Spec}(C)$ the extension $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \to C_\mathfrak{q}/\mathfrak{q}C_\mathfrak{q}$ is purely inseparable, where $\mathfrak{p} = \mathfrak{q} \cap A$.

If ${}_B^*A = A$, A is said to be weakly normal in B.

Now we recall another definition.

Definition 2 (cf. Reid et al. [3, 1.4]). Let $A \subset B$ be an extension of rings. An element $b \in B$ is said to be quasisubintegral over A if there exist $c_1, \ldots, c_p \in B$ such that $b^n + \binom{n}{1}c_1 b^{n-1} + \cdots + \binom{n}{p}c_p b^{n-p} \in A$ for all $n \geq 0$.

Remark 3. Let $A \subset B$ be an extension of rings. Then by [3, 6.10], $b \in {}_B^*A$ if and only if b is quasisubintegral over A.

Definition 4. Let A be a ring and let t be an indeterminate. In the ring $A[[t]]$, let $*$ denote the multiplication $t^n * t^m = \binom{n+m}{n} t^{n+m}$ for all $n, m \geq 0$. Then $(A[[t]], +, *)$ is a ring with 1 as unity.

Lemma 5. *Let A be a ring and let X be an indeterminate. Let $b, c \in A[[X]]$ and let b_0 be the constant term of b. Then for each $r \geq 0$ there exist $d_0, d_1, \ldots, d_r \in A$ such that the coefficient of X^r in cb^n is $d_0 b_0^n + \binom{n}{1} d_1 b_0^{n-1} + \cdots + \binom{n}{r} d_r b_0^{n-r}$ for all $n \geq 0$.*

Proof. Let a_{nr} denote the coefficient of X^r of cb^n for all $r, n \geq 0$.

Let t be an indeterminate and let $*$ denote the multiplication in $A[[X]][[t]]$ as given in Definition 4.

Write $b = b_0 + Xb'$. Then

$$\begin{aligned} c\sum_{n\geq 0} b^n t^n &= c\sum_{n\geq 0}(b_0 + Xb')^n t^n \\ &= \left\{\sum_{n\geq 0} b_0^n t^n\right\} * \left\{\sum_{n\geq 0} cb'^n X^n t^n\right\} \\ &= \left\{\sum_{n\geq 0} b_0^n t^n\right\} * \left\{\sum_{r\geq 0} a_r(t) X^r\right\}, \end{aligned}$$

where, for all $r \geq 0$, $a_r(t) \in A[t]$ and is of degree less than or equal to r.

Comparing the coefficients of X^r in the above expression we get $\sum_{n\geq 0} a_{nr} t^n = \{\sum_{n\geq 0} b_0^n t^n\} * a_r(t)$. Let $a_r(t) = \sum_{i=0}^r d_i t^i$, where $d_i \in A, i = 0, 1, \ldots, r$. Then $a_{nr} = d_0 b_0^n + \binom{n}{1} d_1 b_0^{n-1} + \cdots + \binom{n}{r} d_r b_0^{n-r}$ for all $n \geq 0$. □

Theorem 6. *Let $A \subset B$ be an extension of rings and let X be an indeterminate. Then*

(a) ${}_{B[[X]]}^*(A[[X]]) \subset ({}_B^*A)[[X]]$.

(b) *If ${}_B^*A$ is finite over A then ${}_{B[[X]]}^*(A[[X]]) = ({}_B^*A)[[X]]$. In particular, if A is Noetherian and B is a finite A-module then ${}_{B[[X]]}^*(A[[X]]) = ({}_B^*A)[[X]]$.*

Proof. First note that, by Remark 3, ${}_B^*A \subset {}_{B[[X]]}^*(A[[X]])$. Let $b \in {}_{B[[X]]}^*(A[[X]])$. Write $b = \sum_{i\geq 0} b_i X^i$. By induction on r we show that $b_r \in {}_B^*A$ for all $r \geq 0$. By Remark 3,

b is quasisubintegral over $A[[X]]$. Therefore, there exist $c_1, \ldots, c_p \in B[[X]]$ such that

$$b^n + \sum_{i=1}^{p} \binom{n}{i} c_i b^{n-i} \in A[[X]]$$

for all $n \geq 0$. Putting $X = 0$ we see that b_0 is quasisubintegral over A, i.e. $b_0 \in {}_B^*A$.

Now assume $r \geq 1$ and $b_i \in {}_B^*A$ for all $i < r$. Again by Remark 3, $X^r b'$ is quasisubintegral over $A[[X]]$ where $b' = b_r + b_{r+1}X + \cdots$. Hence, there exist $f_1, \ldots, f_q \in B[[X]]$ such that

$$X^{nr} b'^n + \sum_{i=1}^{q} \binom{n}{i} f_i X^{(n-i)r} b'^{n-i} \in A[[X]]$$

for all $n \geq 0$. Let D_n be the coefficient of X^{nr} in this expression and let D_{in} be the coefficient of X^{ri} in $f_i b'^n$, $i = 1, \ldots, q$. Then $D_n = b_r^n + \sum_{i=1}^{q} \binom{n}{i} D_{i(n-i)}$. Since $D_n \in A$, by Lemma 5 and the identity $\binom{n}{i}\binom{n-i}{j} = \binom{i+j}{j}\binom{n}{i+j}$, we see that b_r is quasisubintegral over A. Therefore, by Remark 3, $b_r \in {}_B^*A$. This proves (a).

To prove (b), let $b_1, \ldots, b_s$ be the generators of ${}_B^*A$ as an A-module. Then $({}_B^*A)[[X]] = \sum_{i=1}^{s} A[[X]] b_i \subset {}_{B[[X]]}^*(A[[X]])$ and hence we have the equality. □

Remark 7. If A is Noetherian and B is a finite extension of A then Theorem 6(b) also follows from [2, Theorem (III.1)] as the extension $A \to A[[X]]$ has geometrically reduced fibres (i.e. for every $\mathfrak{p} \in \operatorname{Spec}(A)$ and for every finite field extension K of $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ the ring $K \otimes_A A[[X]]$ is reduced). However, the condition that ${}_B^*A$ be finite over A is necessary for the equality to hold in Theorem 6 as can be seen from the following example.

Example. Let k be a field of characteristic $p > 0$. Let $Y_1, Y_2, Y_3, \ldots$ be indeterminates and let $B = k[Y_1, Y_2, Y_3, \ldots]$ (resp. $k(Y_1, Y_2, Y_3, \ldots)$) and $A = k[Y_1^p, Y_2^{p^2}, Y_3^{p^3}, \ldots]$ (resp. $k(Y_1^p, Y_2^{p^2}, Y_3^{p^3}, \ldots)$) . Then by [3, 4.3 and 6.10] ${}_B^*A = B$. Let $f = \sum_{n \geq 1} Y_n X^n$. Then for every $r \geq 0$, $f^{p^r} \notin A[[X]]$ and therefore again by [3, 4.3 and 6.10], $f \notin {}_{B[[X]]}^*(A[[X]])$.

However, the equality always holds for polynomial rings.

Corollary 8. *Let A, B and X be as in the Theorem 6. Then ${}_{B[X]}^*(A[X]) = ({}_B^*A)[X]$.*

Proof. It is clear that ${}_B^*A \subset {}_{B[X]}^*(A[X])$ and therefore $({}_B^*A)[X] \subset {}_{B[X]}^*(A[X])$. To prove the other inclusion let $b \in {}_{B[X]}^*(A[X])$. Then, by Remark 3, b is quasisubintegral over $A[X]$ and therefore as an element of $B[[X]]$, b is quasisubintegral over $A[[X]]$. Now by Remark 3 and Theorem 6, $b \in ({}_B^*A)[[X]] \cap B[X] = ({}_B^*A)[X]$. □

Theorem 9. *Let A, B and X be as in Theorem 6. Then the following are equivalent:*

(a) *A is weakly normal in B.*

(b) *$A[[X]]$ is weakly normal in $B[[X]]$.*

(c) *$A[X]$ is weakly normal in $B[X]$.*

Proof. (a)$\Rightarrow$ (b) follows from Theorem 6, (b)$\Rightarrow$ (c) and (c)$\Rightarrow$ (a) are straightforward. □

2. Weak normalisation and the higher derivations

Recall that a higher derivation D of a ring A is a sequence $(D_0, D_1, D_2, \ldots)$ of additive endomorphisms D_n's of A such that D_0 is the identity of A and for all $n \geq 0$ and $a, b \in A$, $D_n(ab) = \sum_{i+j=n} D_i(a)D_j(b)$.

Theorem 10. *Let A be a ring and let K be its total quotient ring. Let $^*A$ be the weak normalisation of A in K. Let D be a higher derivation of A. Then D extends to $^*A$.*

Proof. Write $D = (D_0, D_1, D_2, \ldots)$ and let X be an indeterminate. Then $E : A \to A[[X]]$ defined by $E(a) = \sum_{n \geq 0} D_n(a)X^n$ is a ring homomorphism. Moreover, E extends to $K \to K[[X]]$.

Let $b \in {}^*A$. Then, by Remark 3, b is quasisubintegral over A. Hence, $E(b) \in K[[X]]$ is quasisubintegral over $A[[X]]$. Therefore, by Remark 3 and Theorem 6, $E(b) \in {}^*A[[X]]$, i.e. $D_n(b) \in {}^*A$ for all $n \geq 0$. □

Remark 11. Compare Theorem 10 with [4] and [1, Theorem 4.1].

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References

[1] A. K. Maloo, Maximally differential ideals, J. Algebra 176 (1995) 806–823.
[2] M. Manaresi, Some properties of weakly normal varieties, Nagoya Math. J. 77 (1980) 61–74.
[3] L. Reid, L.G. Roberts and Balwant Singh, On weak subintegrality, preprint.
[4] A. Seidenberg, Derivations and integral closure, Pacific J. Math. 16 (1966) 167–173.
[5] H. Yanagihara, On an intrinsic definition of weakly normal rings, Kobe J. Math. 2 (1985) 89-98.

Note added in proof. After this article was accepted for publication it was brought to my notice by the editor that there is some overlap between the results of this article and the results of the article "Weak normalization of power series rings", Canadian Mathematical Bulletin 38(4) 429–433 by D.E. Dobbs and M. Roitman. For example in that article, Theorem 6(a) and Corollary 8 have been proved for integral domains. However the results of the present article are more general and the methods are entirely different.